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One-Dimensional Chiral Models with First-Order Phase Transitions

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A family of one-dimensional classical chiral spin models with group $G = U(N)$ or $SU(N)$ is introduced, having complex nearest-neighbor interaction. These $G \times G$ invariant systems have self-adjoint positive transfer matrices and satisfy reflection positivity. In the case of $G = U(N)$, for $N = 1, 2, 3$, sequences of first-order phase transitions are shown to occur.

KEY WORDS: Phase transitions; chiral models with groups $U(N)$, $SU(N)$; reflection positivity.

1. INTRODUCTION

One-dimensional classical spin systems having real-valued short-range interaction cannot produce phase transitions. As is well known, phase transitions in such systems only emerge when the restriction on the interaction is appropriately modified. Complex interactions were already considered long ago. Lee and Yang[1] in their general characterization of phase transitions introduced a purely imaginary magnetic field that causes what is now called the Yang–Lee edge singularity. The resulting transfer matrix, however, is not self-adjoint. Investigating the equilibrium equations for the correlation functions of the Ising model, Gallavotti and Lebowitz[2] gave a solution for the one-dimensional system involving a complex Hamiltonian. Recently Asorey and Esteve[3] introduced a family of one-dimensional classical spin models with complex nearest-neighbor interaction, the real part of which constitutes the Z_q clock models. These authors give in the introduction of their work a concise elucidating summary of the conditions for phase transitions of one-dimensional classical systems, to which we

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refer, as well as references. Each of the new models introduced in ref. 3 has self-adjoint transfer matrix, satisfies reflection positivity, and—most interestingly—produces a sequence of first-order phase transitions.

Our work is an extension of ref. 3. We present classical one-dimensional chiral systems with group $G=U(N)$ or $SU(N)$ and complex nearest-neighbor interaction. All the models have the following properties: (i) the interaction is $G\times G$ invariant, (ii) the transfer matrix is a self-adjoint positive trace-class operator on $\mathscr{L}^2(G)$, and (iii) the complex measure generated by the Hamiltonian satisfies reflection positivity. These properties are derived in Section 2 by harmonic analysis. In the case of the group $U(N)$ the character expansion of the transfer matrix is explicitly worked out. The thermodynamic behavior is analyzed in Section 3. We first show for the $U(1)$ model, already treated numerically in ref. 3, an infinite sequence of first-order phase transitions. Furthermore, for large β the transition points and the jumps of the energy per spin and of an order parameter are calculated. For the cases $G=U(2)$ and $G=U(3)$ the existence of sequences of first-order transitions is shown if the imaginary part γ of the coupling is close in strength to the real part. In the complementary region similar behavior is found numerically. Moreover, the $U(2)$ model can be treated analytically in the full range of γ, provided some technical assumptions are accepted. In an appendix we derive inequalities involving modified Bessel functions instrumental in our analysis.

We conclude this introductory part by facing the question of how a complex Hamiltonian might arise physically in the context of classical statistical mechanics. One instance is provided by lattice gauge theories with a topological Θ term included in the interaction. In the author's opinion the one-dimensional systems presented should be considered as simple models which involve complex interactions but nevertheless imply self-adjoint positive transfer matrices.

2. CHIRAL SPIN MODELS WITH COMPLEX INTERACTION

We consider a one-dimensional classical spin chain $\mathsf{X}_{l=1}^{L} G_l$ consisting of L copies of a given unitary group G, where G is either $U(N)$ or $SU(N)$. Denoting by u the standard representation of G by $N\times N$ matrices, we assume the Hamiltonian to be

$$H_L=-\sum_{l=0}^{L-1}\{\mathrm{Re}\ \mathrm{trace}(u_l u_{l+1}^*)+i\gamma\ \mathrm{Im}\ \mathrm{trace}(u_l u_{l+1}^*)\} \tag{2.1}$$

The unit of the inverse temperature β is chosen such that the ferromagnetic coupling constant of the real part is equal to one. The coupling constant

γ of the imaginary part is restricted to the interval $\gamma \in (-1, 1) \subset \mathbb{R}$. Moreover, $u_0 \equiv u_L$, i.e., periodic boundary conditions are imposed. The partition function of the system is

$$Z_L = \int \left(\prod_{l=1}^{L} du_l \right) \exp(-\beta H_L) \tag{2.2}$$

where du denotes the normalized Haar measure on G. The Hamiltonian (2.1) implies the transfer matrix

$$T(u, v) = \exp\{\beta \operatorname{Re} \operatorname{trace}(uv^*) + i\beta\gamma \operatorname{Im} \operatorname{trace}(uv^*)\} \tag{2.3}$$

Viewed as a function of the variable uv^*, (2.3) is a continuous class function on G. Hence a character expansion

$$T(u, v) = \sum_{\tau \in \Omega} d_\tau c_\tau(\beta, \gamma)\, \chi_\tau(uv^*) \tag{2.4}$$

can be performed: the index τ labels the classes of continuous inequivalent unitary irreducible representations of G, and Ω is the family of all such classes; moreover, $d_\tau = \chi_\tau(e)$, where e is the unit element of G. The characters satisfy the generalized orthogonality relations

$$\int du\, \chi_\sigma(uv)\, \overline{\chi_\tau(uw)} = \frac{1}{d_\sigma}\, \delta_{\sigma\tau} \chi_\sigma(vw^*) \tag{2.5}$$

Proposition 1. The class function (2.4) is of positive type, i.e., $c_\tau(\beta, \gamma)$ is real nonnegative, $\forall \tau \in \Omega$.

Proof. Let s label the standard representation of G and $\bar{s}$ its complex conjugate; then $\overline{\chi_s} = \chi_{\bar{s}}$. By series expansion, (2.3) can be written as

$$T(u, v) = \sum_{l=0}^{\infty} \frac{1}{l!} \left(\beta \frac{1+\gamma}{2} \right)^l \sum_{n=0}^{\infty} \frac{1}{n!} \left(\beta \frac{1-\gamma}{2} \right)^n [\chi_s(uv^*)]^l\, [\chi_{\bar{s}}(uv^*)]^n \tag{2.6}$$

Since $1 \pm \gamma > 0$ and all finite-dimensional unitary representations of a compact group are completely reducible (e.g., ref. 4),

$$\chi_\tau(u)\, \chi_\sigma(u) = \sum_{\rho \in \Omega} n^\rho_{\tau\sigma} \chi_\rho(u) \tag{2.7}$$

where $n^\rho_{\tau\sigma}$ are nonnegative integers, Proposition 1 follows.

Proposition 2. The transfer matrix (2.3) considered as an (integral) operator on $\mathscr{L}^2(G)$ is a self-adjoint positive trace-class operator:

(i) $\overline{T(u, v)} = T(v, u)$

(ii) $\int du \int dv\, \bar{h}(u)\, T(u, v)\, h(v) \geqslant 0, \qquad h \in \mathscr{L}^2(G)$

(iii) $\int du\, |T(u, u)| < \infty$

Proof. (i) and (iii) are easily seen and (ii) follows from the expansion (2.4) and Proposition 1.

Comment. $\{c_\tau(\beta, \gamma)\}_{\tau \in \Omega}$ are the eigenvalues of T.

From (2.4) the real, positive series

$$Z_L = \sum_{\tau \in \Omega} d_\tau^2 [c_\tau(\beta, \gamma)]^L \tag{2.8}$$

follows for the partition function (2.2). On the configuration space the Hamiltonian (2.1) gives rise to the normalized complex measure

$$\langle \cdot \rangle_L := (Z_L)^{-1} \int \prod_{l=1}^{L} du_l (\cdot) \exp(-\beta H_L) \tag{2.9}$$

To formulate reflection positivity (e.g., ref. 5) we first choose L even. (i) Reflection at a point between sites amounts to separating the sites of the circular lattice (periodic b.c.) into two disjoint sets by bisecting antipodal bonds: $\Lambda = \Lambda_+ \cup \Lambda_-$ with $\Lambda_+ \equiv \{1, 2,..., L/2\}$, $\Lambda_- \equiv \{L/2+1,..., L\}$. Reflection ϑ of the site n is defined by $\vartheta n = L+1-n$; hence $\vartheta\Lambda_\pm = \Lambda_\mp$, and the related Osterwalder–Schrader involution Θ is

$$\Theta[F(\{u_n\})] = \overline{F(\{u_{\vartheta n}\})} \tag{2.10}$$

The transfer matrix (2.3) satisfies

$$\Theta[T(u_n, u_{n+1})] = T(u_{\vartheta(n+1)}, u_{\vartheta n}) \tag{2.11}$$

(ii) Reflection at a site, say $n = L/2$. Then we decompose $\Lambda = \Lambda_+ \cup \Lambda_0 \cup \Lambda_-$, where $\Lambda_+ \equiv \{1, 2,..., L/2-1\}$, $\Lambda_0 \equiv \{L/2, L\}$, $\Lambda_- \equiv \{L/2+1,..., L-1\}$. Reflection ϑ of site n is defined by $\vartheta n = L-n$, implying $\vartheta\Lambda_\pm = \Lambda_\mp$, $\vartheta\Lambda_0 = \Lambda_0$ and (2.10), (2.11) remain unaltered.

Proposition 3. For a complex function F supported on the configuration space of Λ_+ in the case of (i) (reflection point between sites) or of $\Lambda_+ \cup \Lambda_0$ in the case of (ii) (reflection at a site), respectively, the normalized complex measure (2.9), L even, satisfies

$$\langle F\Theta[F] \rangle_L \in \bar{\mathbb{R}}_+$$

(reflection positivity).

Remark 1. In the case of (ii), reflection positivity is satisfied for $\gamma \in \mathbb{R}$, i.e., outside the interval $(-1, 1)$, too, where the transfer matrix is still self-adjoint, but no longer positive.

Proof. We first consider (i). Because of (2.11) and using for the two couplings connecting Λ_+ with Λ_- the expansion (2.4), together with the factorization

$$\chi_\tau(uv^*) = \sum_{r,s} \mathscr{D}^\tau_{rs}(u)\,\overline{\mathscr{D}^\tau_{rs}(v)} \tag{2.12}$$

in terms of representation matrices, we obtain

$$\langle F\Theta[F]\rangle_L = (Z_L)^{-1} \sum_\sigma d_\sigma c_\sigma \sum_\tau d_\tau c_\tau \sum_{r,s,p,q} \times \left| \int \prod_{l=1}^{L/2} du_l \prod_{n=1}^{L/2-1} T(u_n, u_{n+1})\,\overline{\mathscr{D}^\sigma_{rs}(u_1)}\,\mathscr{D}^\tau_{pq}(u_{L/2})\,F(\{u_m\})\right|^2$$

This form is real, nonnegative.

In the case of (ii), we can write, recalling the convention $u_0 \equiv u_L$,

$$\langle F\Theta[F]\rangle_L = (Z_L)^{-1} \int du_0\, du_{L/2} \left| \int \prod_{l=1}^{L/2-1} du_l \prod_{n=0}^{L/2-1} T(u_n, u_{n+1})\,F(\{u_m\})\right|^2$$

The r.h.s. is obviously real nonnegative, implying Remark 1, too.

Remark 2. Let L be odd. Cutting the circle into two congruent pieces entails reflection at a site and bisecting the opposite bond. In the Hamiltonian the contribution of this bond has to be treated as the bisected bonds in the case (i) of even L, requiring $\gamma \in (-1, 1)$. Then reflection positivity follows by proceeding analogously to the case of even L.

Thermodynamic quantities emerge in the thermodynamic limit: from (2.8) there follows for the free energy f per spin

$$-\beta f = \max_{\sigma \in \Omega} \ln c_\sigma(\beta, \gamma) \tag{2.13}$$

and for the internal energy per spin

$$\varepsilon \equiv -\lim_{L\to\infty} \frac{1}{L}\frac{\partial}{\partial\beta} \ln Z_L = \frac{\partial}{\partial\beta}(\beta f) \tag{2.14}$$

As a local order parameter we introduce the purely imaginary function

$$\eta_l = i\,\mathrm{Im}\,\mathrm{trace}(u_l u^*_{l+1}) \tag{2.15}$$

Due to translation invariance we can use

$$\langle \eta \rangle \equiv \lim_{L \to \infty} \frac{1}{L\beta} \frac{\partial}{\partial \gamma} \ln Z_L = -\frac{\partial}{\partial \gamma} f \tag{2.16}$$

This expectation value of a purely imaginary function is real. The coefficients c_σ determining f, (2.13), are obtained from (2.3), (2.4),

$$d_\sigma c_\sigma(\beta, \gamma) = \int du\, \overline{\chi_\sigma(u)} \exp \beta \{\text{Re trace } u + i\gamma \text{ Im trace } u\} \tag{2.17}$$

For the unitary groups $U(N)$ the characters have been given by Weyl[6]: The classes of continuous inequivalent unitary irreducible representations are characterized by

$$\Omega \ni \sigma \triangleq \{\lambda\} \equiv (\lambda_1, \lambda_2,..., \lambda_N), \qquad \lambda_i \in \mathbb{Z}, \qquad \lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_N \tag{2.18}$$

Class functions on $U(N)$ can be parametrized by $z_l = \exp i\varphi_l$, $\varphi_l \in [0, 2\pi)$, for $l = 1, 2,..., N$; then

$$\text{trace } u = \sum_{j=1}^{N} z_j \tag{2.19}$$

We define according to (2.18) for a given $\{\lambda\}$ the $N \times N$ matrix

$$\mathscr{M}_{kl} = (z_k)^{\lambda_l + N - l} \tag{2.20}$$

with the determinant

$$\Delta_{\{\lambda\}}(z) \equiv \det\{\mathscr{M}_{kl}\}, \qquad \Delta(z) \equiv \Delta_{(0,0,...,0)}(z) \tag{2.21}$$

Then Weyl's famous character formula and the dimension of the representation can be written as

$$\chi_{\{\lambda\}}(z) = [\Delta(z)]^{-1} \Delta_{\{\lambda\}}(z) \tag{2.22a}$$

$$d_{\{\lambda\}} = \left[\prod_{j<k} (k-j)\right]^{-1} \prod_{j<k} (\lambda_j - \lambda_k + k - j) \tag{2.22b}$$

respectively. Moreover, the normalized Haar measure for class functions has the form

$$du = \frac{1}{N!} \left(\prod_{j=1}^{N} \frac{d\varphi_j}{2\pi}\right) |\Delta(z)|^2 \tag{2.23}$$

From (2.22), (2.23) we deduce the coefficients (2.17) as

$$d_{\{\lambda\}}c_{\{\lambda\}}(\beta, \gamma) = \left(\frac{1+\gamma}{1-\gamma}\right)^{(\lambda_1+\lambda_2+\cdots+\lambda_N)/2}$$

$$\times \sum_{\pi \in S_N} \operatorname{sgn} \pi \prod_{j=1}^{N} I_{\lambda_j+\pi(j)-j}(\beta(1-\gamma^2)^{1/2}) \qquad (2.24)$$

Due to Proposition 1 these $N \times N$ determinants formed of modified Bessel functions I_m, $m \in \mathbb{Z}$, are nonnegative! Because of the symmetry relations

$$c_{(\lambda_1, \lambda_2,\ldots,\lambda_N)}(\beta, -\gamma) = c_{(-\lambda_N,\ldots,-\lambda_1)}(\beta, \gamma) \qquad (2.25)$$

it suffices to consider the case $0 \leqslant \gamma < 1$. Moreover, the notation

$$\rho = \left(\frac{1+\gamma}{1-\gamma}\right)^{1/2} > 1, \qquad x = \beta(1-\gamma^2)^{1/2} \qquad (2.26)$$

is used in the sequel.

3. FIRST-ORDER PHASE TRANSITIONS

A. We first treat the case $G = U(1)$. Then (2.24) reduces to

$$n \in \mathbb{Z}: \quad c_n(\beta, \gamma) = \rho^n I_n(x) \qquad (3.1)$$

already obtained in ref. 3. Defining

$$r_n = \ln c_n(\beta, \gamma) \qquad (3.2)$$

we see from (2.13) that we have to look for

$$\hat{r} = \max_{n \in \mathbb{Z}} r_n \qquad (3.3)$$

with γ fixed, as a function of β. We first realize that Lemma A.1 implies $r_n - r_0 < 0$ for n negative. We present a systematic analytic treatment based on the following result.

Proposition 4. Let $n \in \mathbb{N}_0$, $\rho > 1$, and $x \in \mathbb{R}_+$; then

$$I_n(x) \stackrel{!}{=} \rho I_{n+1}(x)$$

has a unique solution $x = x_{n+1}$; it satisfies

$$I_{n+p}(x_{n+1}) > \rho I_{n+p+1}(x_{n+1}), \qquad \forall p \in \mathbb{N}$$

Proof. From Lemma A.1 of the Appendix follow directly the existence and uniqueness of the solution. Lemma A.2 and the first part of the proposition give

$$[I_{n+1}(x_{n+1})]^2 > I_{n+2}(x_{n+1})\, I_n(x_{n+1}) = I_{n+2}(x_{n+1})\, \rho I_{n+1}(x_{n+1})$$

which is the second part for $p=1$. The general case follows by induction. It is useful to consider the difference

$$r_{n+1} - r_n = \ln\left\{\rho \frac{I_{n+1}(x)}{I_n(x)}\right\} \tag{3.4}$$

as a function of increasing x.

0. In the high-temperature region $x \to 0$ we obviously have $\hat{r} = r_0$.
1. Increasing x, because of Lemma A.1 there is a unique x_1 such that

$$I_0(x_1) = \rho I_1(x_1) \tag{3.5}$$

Furthermore, $r_1 \gtrless r_0$ for $x \gtrless x_1$ and from (3.4) we deduce, employing Proposition 4, that

$$x = x_1: \quad r_1 > r_2 > r_3 > \cdots \tag{3.6}$$

2. Increasing $x > x_1$ leads, because of Lemma A.1, to a unique x_2 satisfying

$$I_1(x_2) = \rho I_2(x_2) \tag{3.7}$$

Moreover, $r_2 \gtrless r_1$ for $x \gtrless x_2$ and by Proposition 4

$$x = x_2: \quad r_2 > r_3 > r_4 > \cdots \tag{3.8}$$

Continuing in this manner, we deduce an increasing sequence $\{x_n\}$, $x_0 \equiv 0$, together with

$$n \in \mathbb{N}_0: \quad \hat{r} = r_n \quad \text{for} \quad x_n < x < x_{n+1} \tag{3.9}$$

At each x_n, $n \in \mathbb{N}$, a first-order phase transition occurs, since $\hat{r}$ jumps from one function to another, with (2.14) discontinuous. The transition points x_n for large n can be evaluated asymptotically coupling x with ν by

$$x = a\nu + b + \mathcal{O}\left(\frac{1}{\nu}\right) \tag{3.10}$$

with constants a, b to be determined by solving

$$\frac{1}{\rho} = \frac{I_{\nu+1}(x)}{I_\nu(x)} \tag{3.11}$$

for large ν. Using (A.1), which can be read as the expectation of t with respect to a probability measure, a saddle point expansion yields for (3.10)

$$x \sim \frac{2\rho}{\rho^2-1}\left\{\nu+\frac{\rho^2}{\rho^2+1}+\mathcal{O}\left(\frac{1}{\nu}\right)\right\} \tag{3.12}$$

Observing our enumeration of the transition points x_n,

$$I_{n-1}(x_n)=\rho I_n(x_n) \tag{3.13}$$

we obtain from (3.12) with (2.26)

$$\beta_n \sim \frac{1}{\gamma}\left\{n-\frac{1-\gamma}{2}+\mathcal{O}\left(\frac{1}{n}\right)\right\} \tag{3.14}$$

The discontinuities of the internal energy per spin (2.14) and of the order parameter (2.16) at the nth transition point β_n

$$(\Delta\varepsilon)_n=\frac{2n-1}{\beta_n}-2\gamma \tag{3.15}$$

$$(\Delta\langle\eta\rangle)_n=\frac{1}{1-\gamma^2}\left\{\frac{1+(2n-1)\gamma}{\beta_n}-2\gamma^2\right\} \tag{3.16}$$

behave for large n, because of (3.14), as

$$(\Delta\varepsilon)_n \sim -\frac{\gamma^2}{n}+\mathcal{O}(n^{-2}) \tag{3.17}$$

$$(\Delta\langle\eta\rangle)_n \sim \frac{\gamma}{n}+\mathcal{O}(n^{-2}) \tag{3.18}$$

We observe that both jumps vanish in the limit $\beta_n \to \infty$.

B. Considering now $G=U(2)$, we have the coefficients

$$\begin{gathered}\lambda_1,\lambda_2\in\mathbb{Z}, \qquad \lambda_1\geqslant\lambda_2\\ c_{(\lambda_1,\lambda_2)}(\beta,\gamma)=\rho^{\lambda_1+\lambda_2}I_{\lambda_1,\lambda_2}(x)\end{gathered} \tag{3.19}$$

where the functions $I_{\mu,\nu}$ defined in (A.4) of the Appendix are used. According to (2.13), the thermodynamics is determined by

$$\hat{s}=\max_{\lambda_1\geqslant\lambda_2}\ln c_{(\lambda_1,\lambda_2)} \tag{3.20}$$

The search of the maximum on the two-dimensional array is reduced to a one-dimensional problem by Lemma A.3: it implies that for fixed $\lambda_1+\lambda_2$

the largest coefficient is obtained when $\lambda_1 - \lambda_2$ is smallest. In addition we have the symmetries, for $n \in \mathbb{N}_0$ and $x \in \mathbb{R}_+$,

$$I_{-n,-n} = I_{n,n}, \qquad I_{-n,-n-1} = I_{n+1,n} \tag{3.21}$$

Hence (3.20) is equivalent to

$$\hat{s} = \ln \max_{n \in \mathbb{N}_0} \{c_{(n,n)}, c_{(n+1,n)}\} \tag{3.22}$$

We first analyze (3.22) for large, fixed ρ. We rescale β again, defining

$$z = \rho \frac{x}{2} = \frac{1}{2}(1+\gamma)\beta \tag{3.23}$$

and obtain from Lemma A.5

$$c_{(n,n)} = \frac{z^{2n}}{(n+1)!\,n!}\left\{1 + \frac{2}{n+2}\left(\frac{z}{\rho}\right)^2 f_n\left(\frac{2z}{\rho}\right)\right\} \tag{3.24}$$

$$c_{(n+1,n)} = \frac{z^{2n+1}}{(n+2)!\,n!}\left\{1 + \frac{2}{n+2}\left(\frac{z}{\rho}\right)^2 g_n\left(\frac{2z}{\rho}\right)\right\} \tag{3.25}$$

with f_n and g_n satisfying uniformly in n

$$\frac{14}{15} < f_n\left(\frac{2z}{\rho}\right), g_n\left(\frac{2z}{\rho}\right) < \exp\left(\frac{z}{\rho}\sqrt{2}\right)^2 \tag{3.26}$$

Thereupon we can deduce $\hat{s}$ for fixed ρ sufficiently large and small z. It is then easily seen that for given z no coefficient $c_{(n+1,n)}$ is the largest coefficient of the set entering (3.22). Which of the other coefficients is maximal depends on z. We find first-order phase transitions, $n = 1, 2, 3,...,$

$$\beta_{n-1} < \beta < \beta_n: \quad \hat{s} = \ln c_{(n-1,n-1)} \tag{3.27}$$

with $\beta_0 \equiv 0$ and inverse transition temperatures

$$\beta_n = \frac{2}{1+\gamma}[n(n+1)]^{1/2}\left\{1 + \frac{2}{n+2}\mathcal{O}\left(n(n+1)\frac{1-\gamma}{1+\gamma}\right)\right\} \tag{3.28}$$

The related jumps of the internal energy per spin are

$$\varepsilon(\beta_n + 0, \gamma) - \varepsilon(\beta_n - 0, \gamma) = -\frac{2}{\beta_n} \tag{3.29}$$

The sequence of first-order phase transitions deduced for large values of ρ also appears for small values of ρ. As a typical representative of numerical evaluations performed with small ρ we present in Fig. 1 the case of $\rho=2$, where eight subleading coefficients are included. If one accepts the conjectures stated in the Appendix, the case of $G=U(2)$ can be treated in complete analogy to the analysis performed before with $G=U(1)$: Because of Conjecture 3 of the Appendix, the coefficients $c_{(n+1,n)}$ are irrelevant for $\hat{s}$. Conjecture 2 implies the analogue of Proposition 4, replacing there ρ by ρ^2 and each I_m appearing by $I_{m,m}$. Finally, Conjecture 1 is the substitute of Lemma A.1. As a consequence, an infinite sequence of first-order transitions emerges.

C. For $G=U(3)$ the coefficients (2.24) are

$$\begin{gathered} l, m, n \in \mathbb{Z}, \qquad l \geqslant m \geqslant n \\ c_{(l,m,n)} = \frac{2\rho^{l+m+n}}{(l-m+1)(l-n+2)(m-n+1)} \begin{vmatrix} I_l & I_{l+1} & I_{l+2} \\ I_{m-1} & I_m & I_{m+1} \\ I_{n-2} & I_{n-1} & I_n \end{vmatrix} \end{gathered} \tag{3.30}$$

the modified Bessel functions having the variable x. Similar to $G=U(2)$, we can deal analytically with the case of large, fixed ρ using the rescaling (3.23). For small β the largest coefficient is $c_{(0,0,0)}$. Increasing β, at

$$\beta_1 = \frac{2}{1+\gamma}(6)^{1/3}\{1+\mathcal{O}(1-\gamma)\} \tag{3.31}$$

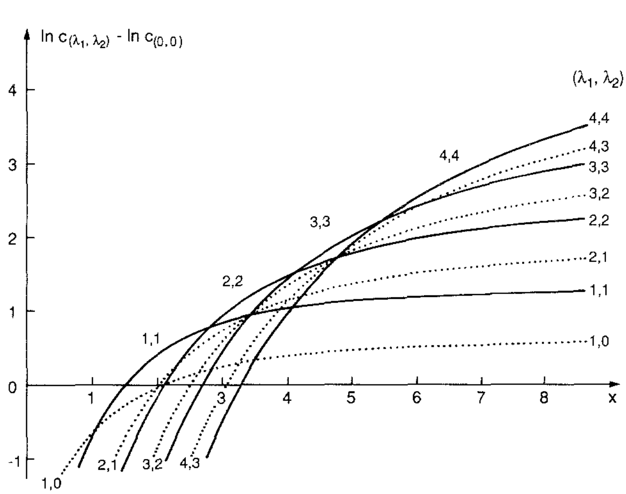

Fig. 1. The case $G=U(2)$ with $\rho=2$: the difference $\ln c_{(\lambda_1,\lambda_2)} - \ln c_{(0,0)}$ for the index pairs (λ_1, λ_2) as indicated is given as a function of x, Eq. (2.26).

$c_{(1,1,1)}$ crosses $c_{(0,0,0)}$ to become the largest coefficient until

$$\beta_2 = \frac{2}{1+\gamma}(24)^{1/3}\{1+\mathcal{O}(1-\gamma)\} \tag{3.32}$$

where $c_{(2,2,2)}$ crosses $c_{(1,1,1)}$ and becomes the largest coefficient. We observe that the order term is n dependent; compare (3.28). Hence, continuing in this way, we eventually reach values of β beyond which the method is no longer applicable. Numerical evaluation of (3.30), however, suggests that these first-order phase transitions occur for all values of ρ.

4. CONCLUDING REMARKS

Within the family of chiral spin models introduced we have shown the existence of sequences of first-order phase transitions for those with $G = U(1)$, $U(2)$, $U(3)$. It seems not too audacious to guess infinite sequences of first-order phase transitions to occur for all $U(N)$ models. It should be noted, however, that the existence of these phase transitions depends on the choice of the boundary conditions: replacing the periodic ones by open boundary conditions, the sum (2.8) reduces to a single term (the coefficient of the trivial representation). It is remarkable that in the non-Abelian cases analyzed only one-dimensional representations (i.e., powers of the determinant of u) survive the thermodynamic limit. For the systems with $G = SU(N)$, $N > 2$, the thermodynamic properties still have to be worked out. Preliminary results indicate that, in contrast to the systems with $G = U(N)$, the group $G = SU(3)$ does not lead to a phase transition; further work is in progress. We conclude by observing that in place of the standard representations used in the Hamiltonian we could employ any other one—even linear combinations with positive coefficients—keeping Propositions 1–3.

APPENDIX. INEQUALITIES INVOLVING MODIFIED BESSEL FUNCTIONS

Here we mainly exhibit inequalities involving modified Bessel functions $I_\nu(x)$ which are instrumental in our analysis.

Lemma A.1. Let $\mathbb{R} \ni \nu > -1/2$, fixed; then the map

$$\frac{I_{\nu+1}}{I_\nu}: \quad \mathbb{R}_+ \to (0, 1)$$

is strictly increasing and bijective.

Proof. A well-known integral representation [ref. 7, Eq. 3.71, (9)] implies

$$\frac{I_{\nu+1}(x)}{I_\nu(x)} = \frac{\int_{-1}^{1} dt\,(1-t^2)^{\nu-1/2} t\,\exp(xt)}{\int_{-1}^{1} dt\,(1-t^2)^{\nu-1/2}\exp(xt)} \tag{A.1}$$

Hence, we can write

$$\frac{I_{\nu+1}(x)}{I_\nu(x)} = \frac{d}{dx}\ln g(x) \tag{A.2}$$

$$g(x) = \int_{-1}^{+1} dt\,(1-t^2)^{\nu-1/2}\exp(xt) \tag{A.3}$$

Due to its particular form, $g(x)$ is "log-convex," i.e., $\ln g(x)$ is strictly convex (e.g., ref. 8), hence its derivative is strictly increasing. The range of the map follows by considering the quotient at small and large values of $x \in \mathbb{R}_+$.

Lemma A.2. Let $m \in \mathbb{N}$ and $x \in \mathbb{R}_+$:

$$[I_m(x)]^2 > I_{m+1}(x)\,I_{m-1}(x)$$

This is proven later.

We define for $\mu, \nu \in \mathbb{R}$, $\mu \geqslant \nu$, and $x \in \mathbb{R}_+$

$$I_{\mu,\nu}(x) = (\mu-\nu+1)^{-1}\{I_\mu(x)\,I_\nu(x) - I_{\mu+1}(x)\,I_{\nu-1}(x)\} \tag{A.4}$$

From ref. 7, Eq. 5.41, (1), there follows the power series representation

$$I_{\mu,\nu}(x) = \left(\frac{x}{2}\right)^{\mu+\nu}\sum_{k=0}^{\infty}\left(\frac{x}{2}\right)^{2k}\frac{\Gamma(\mu+\nu+2k+1)}{k!\,\Gamma(\mu+\nu+k+1)\,\Gamma(\mu+k+2)\,\Gamma(\nu+k+1)} \tag{A.5}$$

If $\mu+\nu$ is a negative integer, the quotient of the Γ functions with $\mu+\nu$ in the respective arguments has to be represented by the product $(\mu+\nu+k+1)_k$; see, e.g., ref. 7, Section 4.4. Considering (A.5) at $\mu=\nu=m\in\mathbb{N}$ immediately proves Lemma A.2. Moreover, (A.5) with $\mu,\nu\in\mathbb{Z}$ proves directly the positivity of the coefficients of the character expansion (2.4) in the case of $G=U(2)$, a result inferred before more generally from the representation theory of compact groups.

Lemma A.3. For $m, n \in \mathbb{Z}$, $m \geqslant n$, and $x \in \mathbb{R}_+$

$$I_{m,n}(x) > I_{m+1,n-1}(x)$$

Proof. From (A.5) there follows for the difference

$$\begin{aligned}
I_{m,n}(x) - I_{m+1,n-1}(x) \\
&= (m-n+2)\left(\frac{x}{2}\right)^{m+n} \sum_{k=0}^{\infty} \left(\frac{x}{2}\right)^{2k} \\
&\quad \times \frac{\Gamma(m+n+2k+1)}{k!\,\Gamma(m+n+k+1)\,\Gamma(m+k+3)\,\Gamma(n+k+1)}
\end{aligned} \tag{A.6}$$

proving the claim.

Lemma A.4. For $n \in \mathbb{N}_0$ and $x \in \mathbb{R}_+$

$$I_{n,n}(x) > I_{n+1,n+1}(x)$$

Proof. Due to Lemma A.1, the x derivative of I_{n+1}/I_n is positive, implying the claim when the functional equation for I'_m is used.

For part of our analysis it is necessary to control $I_{n,n}$ and $I_{n+1,n}$ for given x uniformly in n. From (A.5) we obtain, employing the duplication formula of the Γ function, the following result.

Lemma A.5. For $n \in \mathbb{N}_0$ and $x \in \mathbb{R}_+$

$$I_{n,n}(x) = \frac{(x/2)^{2n}}{(n+1)!\,n!} \left\{1 + \frac{2}{n+2}\left(\frac{x}{2}\right)^2 f_n(x)\right\}$$

$$I_{n+1,n}(x) = \frac{(x/2)^{2n+1}}{(n+2)!\,n!} \left\{1 + \frac{2}{n+2}\left(\frac{x}{2}\right)^2 g_n(x)\right\}$$

$$\frac{14}{15} < f_n(x),\ g_n(x) < \exp\left(\frac{1}{2}x^2\right)$$

uniformly in n.

Finally we state three conjectures which unfortunately we could not prove; they have been tested numerically for $n \leqslant 20$.

Conjecture 1. For $n \in \mathbb{N}_0$, the quotient $[I_{n,n}(x)]^{-1} I_{n+1,n+1}(x)$ maps $\mathbb{R}_+$ on $(0, 1)$ strictly increasing.

Conjecture 2. For $n \in \mathbb{N}$, $x \in \mathbb{R}_+$

$$[I_{n,n}(x)]^2 > I_{n+1,n+1}(x)\, I_{n-1,n-1}(x)$$

Conjecture 3. Let $\mathbb{R}_+ \ni \rho > 1$ and $n \in \mathbb{N}_0$; then

$$\rho I_{n+1,n}(x) \stackrel{!}{=} \rho^2 I_{n+1,n+1}(x) < I_{n,n}(x)$$

for the unique solution x of the equation.

Comment. Conjectures 1 and 2 can be verified analytically for small and large values of x.

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